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O-MINIMAL EXPANSIONS OF REAL CLOSED FIELDS AND COMPLETENESS IN THE SENSE OF SCOTT

OLIVIER FRÉCON

ABSTRACT. We consider an o-minimal expansion $\mathcal{M}_0 = (R_0, <, +, \cdots)$ of a real closed field, and a real closed field R, complete in the sense of D. Scott, containing R_0 as a dense subfield. We show that $\mathcal{M}_0$ has an elementary extension $\mathcal{M} = (R, <, +, \cdots)$ with domain R. Moreover, such a structure $\mathcal{M}$ with domain R is unique.

NOTE

*In an unpublished article, Antongiulio Fornasiero proved a more general result than the main theorem of this paper. Indeed, he showed a similar result for d-*minimal *expansions of a real closed field* [1, Proposition 11.6]*. However our proofs are very different.*

1. INTRODUCTION

By the Compactness Theorem, it is easy to elementarily embed any expansion $\mathcal{M}_0 = (R_0, <, +, \cdots)$ of a real closed field in an expansion $\mathcal{M} = (R, < +, \cdots)$ of a *Scott-complete* real closed field, that is complete in the sense of Dana Scott (Definition 1.1). However, we have little control on the size of the elementary extension obtained. For instance, if R_0 is countable, it is possible that the field R has no countable dense subfield. The main result of this paper shows that any *o-minimal* expansion of a real closed field R_0 is elementarily embedded in an expansion of a Scott-complete real closed field, in which R_0 is dense (Theorem 1.2). We note that, since we consider an expansion of a real closed field, not just the field structure, the model completeness of the theory of real closed fields will not help us.

We recall the definition of a *complete* real closed field in the sense of [2].

Definition 1.1.. – *If K and L are ordered fields and $K \subseteq L$, then K is* dense *in L if between any two distinct elements of L there lies an element of K.*

A given ordered field is called Scott-complete *if it has no proper extension to an ordered field in which the given field is dense.*

The main result of this paper is the following.

Theorem 1.2.. – *Any o-minimal expansion of a real closed field R_0 has an elementary extension of domain a Scott-complete field R in which R_0 is dense. Moreover, for a fixed field R, this elementary extension is unique.*

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2. Notations and preliminary results

For the rest of this paper, we fix an o-minimal expansion $\mathcal{M}_1 = (R_0, <, \cdots)$ of a real closed field R_0, and we denote by R a Scott-complete field in which R_0 is dense: its existence is ensured by Fact 2.1 below.

Fact 2.1.. – [2, Theorem 1] *Given any ordered field K, there is a Scott-complete ordered field $\hat{K}$ in which K is dense. Any other Scott-complete ordered field in which K is dense is isomorphic to $\hat{K}$ by a unique isomorphism that is the identity on K.*

In the following, we say that a set X is *definable* if it is definable in the structure $\mathcal{M}_1$.

For each integer k and each subset A of R^k, we denote by $\overline{A}$ its (topological) closure in R^k. For each subset X of R_0^k, we denote by $\check{X}$ the union of subsets $\overline{F}$ for F a R_0-closed definable subset contained in X.

For clarity, we use a very different notation for the closure in R_0^k of a subset X of R_0^k: we denote it by $\mathrm{cl}\, X$. Moreover, we denote by ∂X its frontier: $\partial X = \mathrm{cl}\, X \setminus X$.

For any element $x = (x_1, \ldots, x_k)$ of R^k, we consider

$$|x| = \max\{|x_i| \mid i \in \{1, \ldots, k\}\}$$

Remark 2.2.. –

- For any subset X of R_0^k, it follows from the definition of $\check{X}$ that $X = \check{X} \cap R_0^k$.
- The proof of Theorem 1.2 will show that the elementary extension of $\mathcal{M}_1$ of domain R has the following property:

  for any two integers k and m, if A is any subset of R_0^k defined by a formula $\varphi(\overline{x}, \overline{a})$ with free variables $\overline{x} = (x_1, \ldots, x_k)$ and parameters $\overline{a} \in R_0^m$, then the subset of R^k defined by $\varphi(\overline{x}, \overline{a})$ is $\check{A}$.

The first lemma is certainly well-known, but we could not find a reference for it.

Lemma 2.3.. – *Let k be an integer, X a subset of R_0^k, and $f : X \to R_0$ be a uniformly continuous map. Then for each $x \in \overline{X}$, the following limit exists:*

$$\lim_{\substack{y \in X \\ y \to x}} f(y)$$

Proof – Let $\varepsilon \in R_0^{>0}$. Since f is uniformly continuous on X, then we may associate with ε some $\delta(\varepsilon) \in R_0^{>0}$ such that whenever $|v - u| < \delta(\varepsilon)$ for u and v in X, we have $|f(v) - f(u)| < \varepsilon$. Then for each $y \in X$ and $z \in X$ such that $|y - x| < \delta(\varepsilon)/2$ and $|z - x| < \delta(\varepsilon)/2$, we have $|z - y| < \delta(\varepsilon)$ and $|f(z) - f(y)| < \varepsilon$.

Let C_ε^- (resp. C_ε^+) be the set of elements $\alpha \in R_0$ such that $\alpha < f(y) - \varepsilon$ (resp. $\alpha > f(y) + \varepsilon$) for some $y \in X$ satisfying $|y - x| < \delta(\varepsilon)/2$. Let C^- (resp. C^+) be the union of subsets of the form C_ε^- (resp. C_ε^+) for $\varepsilon \in R_0^{>0}$.

Claim 1: *for each $a \in C^-$ and each $b \in C^+$, we have $a < b$. In particular, the set $C^- \cap C^+$ is empty.*

There exist y_1 and y_2 in X, and ε_1 and ε_2 in $R_0^{>0}$, such that $|y_1 - x| < \delta(\varepsilon_1)/2$, $|y_2 - x| < \delta(\varepsilon_2)/2$, $a < f(y_1) - \varepsilon_1$ and $b > f(y_2) + \varepsilon_2$. Let $\delta = \min\{\delta(\varepsilon_1), \delta(\varepsilon_2)\}$. We fix $y \in X$ such that $|y - x| < \delta/2$. Then we have $|f(y) - f(y_1)| < \varepsilon_1 < f(y_1) - a$ and $|f(y) - f(y_2)| < \varepsilon_2 < b - f(y_2)$, so we obtain $a < f(y) < b$.

Claim 2: *for each $r \in R_0^{>0}$, there is $a \in C^-$ and $b \in C^+$ such that $b - a < r$.*

Let $y \in X$ such that $|y-x| < \delta(r/3)/2$. Then for $a = f(y) - r/2$ and $b = f(y) + r/2$, we have $a \in C^-_{r/3}$ and $b \in C^+_{r/3}$, and we obtain $a \in C^-$, $b \in C^+$ and $b - a < r$.

Conclusion: By [3, §2, Lemma 2.8] (or [2]), there is a unique $\omega \in R$ satisfying $a < \omega < b$ for every $a \in C^-$ and $b \in C^+$. Hence for each $\varepsilon \in R_0^{>0}$, there is $\delta(= \delta(\varepsilon)) \in R_0^{>0}$ such that whenever $|y-x| < \delta/2$ for $y \in X$, we have $\omega \geq f(y) - \varepsilon$ and $\omega \leq f(y) + \varepsilon$, so $|f(y) - \omega| \leq \varepsilon$. Now $\omega = \lim_{\substack{y \in X \\ y \to x}} f(y)$ exists. □

Lemma 2.4.. – *Let k be an integer. If E and F are two closed definable subsets of R_0^k, then $\overline{E} \cap \overline{F} = \overline{E \cap F}$. In particular, if $\overline{E} \cap \overline{F}$ is non-empty, then $E \cap F$ is non-empty too.*

Proof – We have just to prove that $\overline{E \cap F}$ contains $\overline{E} \cap \overline{F}$. Let $x \in \overline{E} \cap \overline{F}$. For each $\varepsilon \in R_0^{>0}$, we fix $u_\varepsilon \in E$ such that $|u_\varepsilon - x| < \varepsilon$. Let $B_\varepsilon = \{y \in R_0^k \mid |u_\varepsilon - y| \leq \varepsilon\}$. Since $x \in \overline{E} \cap \overline{F}$ and since $|u_\varepsilon - x| < \varepsilon$, we have $x \in \overline{E_\varepsilon} \cap \overline{F_\varepsilon}$ where $E_\varepsilon = B_\varepsilon \cap E$ and $F_\varepsilon = B_\varepsilon \cap F$. Moreover, we note that E_ε and F_ε are closed and bounded definable subsets of R_0^k.

We show that $E_\varepsilon \cap F_\varepsilon$ is non-empty. Let $f_\varepsilon : E_\varepsilon \times F_\varepsilon \to R_0$ defined by $f_\varepsilon(z, z') = |z - z'|$. It is a definable continuous function, so its image is closed and bounded (see [4, Chapter 6 §1]). For each $\eta \in R_0^{>0}$, there exist $u \in E_\varepsilon$ and $v \in F_\varepsilon$ such that $|u - x| < \eta/2$ and $|v - x| < \eta/2$, so we have $f(u,v) = |u - v| < \eta$. Since the image of f_ε is closed and bounded, this implies that it contains zero. Hence there exist $a \in E_\varepsilon$ and $b \in F_\varepsilon$ such that $f_\varepsilon(a,b) = 0$. Now we have $a = b \in E_\varepsilon \cap F_\varepsilon$, and $E_\varepsilon \cap F_\varepsilon$ is non-empty.

Since $|u_\varepsilon - x| < \varepsilon$, we have $|y - x| < 2\varepsilon$ for any $y \in B_\varepsilon$. Thus, the previous paragraph proves that for each $\varepsilon \in R_0^{>0}$, there exists $y \in E \cap F$ such that $|y - x| \leq 2\varepsilon$, so $x \in \overline{E \cap F}$, as desired. □

Corollary 2.5.. – *For each subset X of R_0^k, and each $x \in \breve{X}$, there is a closed and bounded definable subset F of X such that $x \in \overline{F}$.*

Proof – Since $x \in \breve{X}$, there exists $x_0 \in X$ such that $|x - x_0| < 1$. We consider $F_1 = \{y \in R_0^k \mid |y - x_0| \leq 1\}$. Then F_1 is a closed and bounded definable subset of R_0^k. By density of R_0 in R and since $|x - x_0| < 1$, we have $x \in \overline{F_1}$.

Moreover, by the definition of $\breve{X}$, there is a closed definable subset F_2 of X such that $x \in \overline{F_2}$. Then $F = F_1 \cap F_2$ is a closed and bounded definable subset of X, and Lemma 2.4 gives $x \in \overline{F}$. □

Proposition 2.6.. – *Let k be an integer. If $\{A_1, \ldots, A_m\}$ is a partition of R_0^k into definable subsets, then $\{\breve{A}_1, \ldots, \breve{A}_m\}$ is a partition of R^k.*

Proof – First we show that R^k is the union of $\breve{A}_1, \ldots, \breve{A}_m$. Since each definable subset of R_0^k has a decomposition into cells (see [4, Chapter 3 §2] for more details), we may assume that $A_1, \ldots, A_m$ are cells.

Let $x \in R^k$. We show that $x \in \breve{A}_j$ for some $j \in \{1, \ldots, m\}$. We may assume $x \notin R_0^k$. By finiteness of the partition $\{A_1, \ldots, A_m\}$, there exists $r \in R^{>0}$ such that, for any $s \in]0, r]$, the set $I = \{i \in \{1, \ldots, m\} \mid \exists a \in A_i, |x - a| < s\}$ is constant. Since R_0 is dense in R, the set I is non-empty, and by the definition of I, the point x is in the R^k-closure $\overline{A_i}$ of A_i for each $i \in I$.

Let d be the smallest integer such that there is $j \in \{1, \dots, m\}$ and a definable subset B of A_j of dimension d with x contained in $\overline{B}$. Let $B_i = A_i \cap \partial B$ for each $i \in \{1, \dots, m\}$. Then for each $i \in \{1, \dots, m\}$, the subset B_i of A_i is definable (see [4, Chapter 1 §3]) and we have $\dim B_i \leq \dim \partial B < \dim B = d$ [4, Chapter 4 §1]. By the minimality of d, the point x is contained in the R^k-closure $\overline{B_i}$ of B_i for no $i \in \{1, \dots, m\}$. Consequently, there is $t \in R^{>0}$ such that $|y - x| > t$ for any $y \in \cup_{i=1}^m B_i$, and we may choose $t \in R_0$ as R_0 is dense in R. Since $\cup_{i=1}^m B_i = \partial B$ and since $x \in \overline{B}$, there exists $b_0 \in B$ such that $|b_0 - x| < t/2$. Let $B_f = \{b \in B \mid |b - b_0| \leq t/2\}$. By the choices of t and b_0, we have $x \in \overline{B_f}$ and $\partial B \cap B_f = \emptyset$. This implies that the set B_f is a closed definable subset of B and that x belongs to $\breve{A}_j$, as desired.

We show that $\breve{A}_i \cap \breve{A}_j = \emptyset$ for any distinct elements i and j of $\{1, \dots, m\}$. Otherwise there is a closed definable subset F_i (resp. F_j) of A_i (resp. A_j) such that $\overline{F_i} \cap \overline{F_j} \neq \emptyset$. By Lemma 2.4, the set $F_i \cap F_j$ is non-empty, contradicting $A_i \cap A_j = \emptyset$. Thus $\{\breve{A}_1, \dots, \breve{A}_m\}$ is a partition of R^k. □

3. Proof of the main theorem

We provide three preparatory results before the final argument.

Lemma 3.1.. – *Let* $S_0 = \{(u, v) \in R_0^2 \mid u < v\}$ *and* $S = \{(u, v) \in R^2 \mid u < v\}$. *Then we have* $S = \breve{S}_0$.

Proof – First we show that $\breve{S}_0$ contains S. Let $(u, v) \in S$. Then there exists $r \in R_0^{>0}$ such that $v - u > r$. Let $(u_0, v_0) \in R_0^2$ such that $|u - u_0| < r/4$ and $|v - v_0| < r/4$, and let $F = \{(x, y) \in R_0^2 \mid |x - u_0| \leq r/4, |y - v_0| \leq r/4\}$. In particular, we have $(u, v) \in \overline{F}$. Moreover we have $v_0 - u_0 > r/2$, so we obtain $y - x > 0$ for each $(x, y) \in F$, and F is a closed definable subset of S_0. Consequently (u, v) belongs to $\breve{S}_0$, and $\breve{S}_0$ contains S.

Now we show that S contains $\breve{S}_0$. Let $(u, v) \in \breve{S}_0$. By Corollary 2.5, there is a closed and bounded definable subset F_0 of S_0 such that $(u, v) \in \overline{F_0}$. Let $f : F_0 \to R_0$ defined by $f(x, y) = y - x$. Then f is a definable continuous function, so its image is closed and bounded. Let $m = \min\{f(x, y) \mid (x, y) \in F_0\}$. Then we have $b - a \geq m$ for each $(a, b) \in \overline{F_0}$. Since $F_0 \subseteq S_0$, we have $m > 0$ and we obtain $(u, v) \in S$. □

Corollary 3.2.. – *Let* $T_0 = \{(u, v) \in R_0^2 \mid u = v\}$ *and* $T = \{(u, v) \in R^2 \mid u = v\}$. *Then we have* $T = \breve{T}_0$.

Proof – Let $S_0 = \{(u, v) \in R_0^2 \mid u < v\}$ and $S_1 = \{(u, v) \in R_0^2 \mid u > v\}$. Then $\{S_0, S_1, T_0\}$ is a partition of R_0^2, and Proposition 2.6 says that $\{\breve{S}_0, \breve{S}_1, \breve{T}_0\}$ is a partition of R^2. Now Lemma 3.1 gives $\breve{T}_0 = R^2 \setminus (\breve{S}_0 \cup \breve{S}_1) = T$. □

Lemma 3.3.. – *If* G_0 *(resp.* H_0, K_0*) denotes the graph of* $\cdot$ *(resp.* $+$, $-$*) in* R_0^3 *(resp.* R_0^3, R_0^2*), then the graph of* $\cdot$ *(resp.* $+$, $-$*) in* R^3 *(resp.* R^3, R^2*) is* $\breve{G}_0$ *(resp.* $\breve{H}_0$, $\breve{K}_0$*).*

Proof – Since $\cdot$ is a continuous map over R_0, its graph G_0 in R_0^3 is closed, and we have $\breve{G}_0 = \overline{G_0}$. Moreover, since $\cdot$ is a continuous map over R, its graph G in R^3 is closed, and since G contains G_0, it contains $\overline{G_0}$ too. But R_0 is dense in R, hence for each $(x, y) \in R^2$ we have $(x, y, x \cdot y) \in \overline{G_0}$, and G is contained in $\overline{G_0}$. We conclude that $G = \overline{G_0} = \breve{G}_0$, as desired.

In the same way, we show that the graph of $+$ (resp. $-$) in R^3 (resp. R^2) is $\breve{H}_0$ (resp. $\breve{K}_0$). $\square$

Proof of Theorem 1.2 – We denote by $\mathcal{L}_1$ the language of $\mathcal{M}_1 = (R_0, <, \cdots)$. For each function symbol f of $\mathcal{L}_1$ with arity k, we consider a relation symbol S_f such that $S_f^{\mathcal{M}_1}$ is the graph of $f^{\mathcal{M}_1}$, and for each constant symbol c of $\mathcal{L}_1$, we consider a relation symbol S_c such that $S_c^{\mathcal{M}_1} = c^{\mathcal{M}_1}$. We obtain a relational language $\mathcal{L}$ and a structure $\mathcal{M}_0 = (R_0, <, \cdots)$ in $\mathcal{L}$. We have just to prove that there is a unique $\mathcal{L}$-structure $\mathcal{M}$ with domain R such that $\mathcal{M}$ is an elementary extension of $\mathcal{M}_0$.

We note that, for any integer k, a subset X of R_0^k is definable (in $\mathcal{M}_1$) if and only if it is definable in $\mathcal{M}_0$.

Uniqueness:

First we assume that the structure $\mathcal{M}$ exists. Let S be any relation symbol of arity k of $\mathcal{L}$. Let F be a closed and bounded definable subset of $S^{\mathcal{M}_0}$, and let $\varphi(\overline{x}, \overline{a})$ be an $\mathcal{L}$-formula with free variables $\overline{x} = (x_1, \ldots, x_k)$ and parameters $\overline{a} \in R_0^m$ such that F is defined by $\varphi(\overline{x}, \overline{a})$. Let $\tilde{F}$ be the subset of R^k defined by $\varphi(\overline{x}, \overline{a})$. Since $\mathcal{M}$ is an elementary extension of $\mathcal{M}_0$, then $\tilde{F}$ contains F and it is closed and bounded in R^k. Thus $\tilde{F}$ contains $\overline{F}$. By Corollary 2.5, this implies that $S^{\mathcal{M}}$ contains $\breve{S}^{\mathcal{M}_0}$.

In the same way, the complementary of $S^{\mathcal{M}}$ in R^k contains $\widehat{R_0^k \setminus S^{\mathcal{M}_0}}$. Now Proposition 2.6 gives $S^{\mathcal{M}} = \breve{S}^{\mathcal{M}_0}$, so, if it exists such a structure $\mathcal{M}$, then it is unique.

Existence:

We consider the $\mathcal{L}$-structure $\mathcal{M} = (R, <, \cdots)$ where for each relation symbol S of arity k of $\mathcal{L}$, we define $S^{\mathcal{M}}$ by $S^{\mathcal{M}} = \breve{S}^{\mathcal{M}_0}$. By Lemmas 3.1 and 3.3 and Corollary 3.2, it is sufficient to show that $\mathcal{M}$ is an elementary extension of $\mathcal{M}_0$.

First we note that for each relation symbol S of arity k of $\mathcal{L}$, we have $S^{\mathcal{M}_0} = \breve{S}^{\mathcal{M}_0} \cap R_0^k$ (Remark 2.2), so $\mathcal{M}_0$ is a substructure of $\mathcal{M}$.

Claim 1: *if A is a definable subset of R_0^k for an integer k, then we have $\breve{A}_0 = R^k \setminus \breve{A}$ where $A_0 = R_0^k \setminus A$.*

Since $\{A, A_0\}$ is a partition of R_0^k, then Proposition 2.6 says that $\{\breve{A}, \breve{A}_0\}$ is a partition of R^k.

Claim 2: *if A and B are two definable subsets of R_0^k for an integer k, then we have $\breve{A} \cap \breve{B} = \widehat{A \cap B}$.*

It is sufficient to prove that $\widehat{A \cap B}$ contains $\breve{A} \cap \breve{B}$. Let $x \in \breve{A} \cap \breve{B}$. Then there exist a closed definable subset E of A and a closed defnable subset F of B such that $x \in \overline{E} \cap \overline{F}$. Now x belongs to $\overline{E \cap F} \subseteq \widehat{A \cap B}$ by Lemma 2.4, and we obtain $\breve{A} \cap \breve{B} = \widehat{A \cap B}$.

Claim 3: *for any two integers k and l, if A and B are definable subsets of R_0^k and R_0^l respectively, then we have $\breve{A} \times \breve{B} = \widehat{A \times B}$.*

Let $x \in \breve{A} \times \breve{B}$. We show that x belongs to $\widehat{A \times B}$. We have $x = (u, v)$ for $u \in \breve{A}$ and $v \in \breve{B}$. Then there are two closed definable subsets F and G of A and B respectively such that $u \in \overline{F}$ and $v \in \overline{G}$. For each $\varepsilon \in R^{>0}$, there $f \in F$ and

$g \in G$ such that $|u - f| < \varepsilon$ and $|v - g| < \varepsilon$, so $|(u, v) - (f, g)| < \varepsilon$. Consequently, $x = (u, v)$ belongs to $\overline{F \times G} \subseteq \widehat{A \times B}$.

By Corollary 2.5, for any $x \in \widehat{A \times B}$, there is a closed and bounded definable subset H of $A \times B$ such that $x \in \overline{H}$. If H_1 (resp. H_2) denotes the image of H by the projection $\pi_1 : A \times B \to A$ (resp. $\pi_2 : A \times B \to B$), then by the continuity of the projections maps, the set H_1 (resp. H_2) is closed and bounded, and H is contained in $H_1 \times H_2$. Now $\overline{H}$ is contained in $\overline{H_1 \times H_2}$. But $\overline{H_1} \times \overline{H_2}$ is closed in R^{k+l} and it contains $H_1 \times H_2$, so it contains $\overline{H_1 \times H_2}$. Hence $\overline{H}$ is contained in $\overline{H_1} \times \overline{H_2}$, and $\breve{A} \times \breve{B}$ contains $\widehat{A \times B}$.

Claim 4: *let $\pi : R^{k+1} \to R^k$ be the projection on the first k coordinates for an integer k. If A is a definable subset of R_0^{k+1}, then we have $\pi(\breve{A}) = \widehat{\pi(A)}$.*

First we note that the restriction $\pi_{|R_0^{k+1}} : R_0^{k+1} \to R_0^k$ of π to R_0^{k+1} is definable and continuous. In particular, the set $\pi(A)$ is definable.

Let $x \in \breve{A}$. By Corollary 2.5, there is a closed and bounded definable subset F of A such that $x \in \overline{F}$. Then, for each $\varepsilon \in R^{>0}$, there exists $y \in F$ such that $|y - x| < \varepsilon$, so we have $|\pi(y) - \pi(x)| < \varepsilon$, and thus we obtain $\pi(x) \in \overline{\pi(F)}$ But the restriction $\pi_{|R_0^{k+1}}$ is definable and continuous, so $\pi(F)$ is a closed and bounded definable subset contained in $\pi(A)$. Hence $\pi(x)$ belongs to $\widehat{\pi(A)}$, and $\widehat{\pi(A)}$ contains $\pi(\breve{A})$.

Let $x \in \widehat{\pi(A)}$. We show that x belongs to $\pi(\breve{A})$. By definable choice [4, Chapter 6 §1], there is a definable map $f : \pi(A) \to R_0$ such that $\{(\alpha, f(\alpha)) \mid \alpha \in \pi(A)\}$ is contained in A. By cell decomposition [4, Chapter 3 §2], there are finitely many cells $C_1, \ldots, C_s$ of $\pi(A)$ such that $\pi(A) = \cup_{i=1}^s C_i$ and f is continuous on C_i for each i. By Proposition 2.6, we have $x \in \breve{C}_r$ for some $r \in \{1, \ldots, s\}$. By Corollary 2.5, there is a closed and bounded definable subset G of C_r such that $x \in \overline{G}$. But f is continuous on G, so the graph Γ of its restriction $f_{|G} : G \to R_0$ to G is a closed and definable subset of A. Moreover, the continuity of f on G implies its uniform continuity on G [4, Chapter 6 §1], hence the following limit exists by Lemma 2.3:

$$u = \lim_{\substack{y \in G \\ y \to x}} f(y)$$

Now (x, u) belongs to $\overline{\Gamma}$, and since Γ is a closed and definable subset of A, we obtain $(x, u) \in \breve{A}$ and $x \in \pi(\breve{A})$.

Claim 5: *if $\varphi(\overline{x}, \overline{a})$ is an atomic formula with free variables $\overline{x} = (x_1, \ldots, x_k)$ and parameters $\overline{a} = (a_1, \ldots, a_m)$ in R_0^m, and if A is the definable subset of R_0^k defined by $\varphi(\overline{x}, \overline{a})$, then $\breve{A}$ is the $\mathcal{M}$-definable subset of R^k defined by $\varphi(\overline{x}, \overline{a})$.*

Let S be a relation symbol such that $\varphi(\overline{x}, \overline{a}) = S(\overline{x}, \overline{a})$. By the definition of $S^{\mathcal{M}}$, we have $S^{\mathcal{M}} = \breve{S}^{\mathcal{M}_0}$. Therefore, if $B = S^{\mathcal{M}_0}$ is the subset of R_0^{k+m} defined by $\varphi(\overline{x}, \overline{y})$, if $C = R_0^k \times \{\overline{a}\}$, and if $\pi : R^{k+m} \to R^k$ is the projection on the first k coordinates, then we have $A = \pi(B \cap C)$. In the same way, the $\mathcal{M}$-definable subset of R^k defined by $\varphi(\overline{x}, \overline{a})$ is $\pi(S^{\mathcal{M}} \cap (R^k \times \{\overline{a}\}))$. Since by Claim 3 we have $\breve{C} = \breve{R}_0^k \times \widehat{\{\overline{a}\}} = R^k \times \{\overline{a}\}$, and since by Claims 2 and 4 we have $\breve{A} = \pi(\breve{B} \cap \breve{C})$, Claim 5 is proven.

Claim 6: *let $\varphi(\overline{x}, y, \overline{a})$ be a formula with free variables $\overline{x} = (x_1, \dots, x_k)$ and y, and parameters $\overline{a} \in R_0^m$. Let A be the subset of R_0^{k+1} defined by $\varphi(\overline{x}, y, \overline{a})$, and B be the subset of R_0^k defined by $\exists y\, \varphi(\overline{x}, y, \overline{a})$. If the subset of R^{k+1} defined by $\varphi(\overline{x}, y, \overline{a})$ is $\breve{A}$, then the subset of R^k defined by $\exists y\, \varphi(\overline{x}, y, \overline{a})$ is $\breve{B}$.*

This follows from Claim 4.

Claim 7: *let $\varphi(\overline{x}, \overline{a})$ be a formula with free variables $\overline{x} = (x_1, \dots, x_k)$ and parameters $\overline{a} \in R_0^m$. Let A be the subset of R_0^k defined by $\varphi(\overline{x}, \overline{a})$, and B be the subset of R_0^k defined by $\neg\varphi(\overline{x}, \overline{a})$. If the subset of R^k defined by $\varphi(\overline{x}, \overline{a})$ is $\breve{A}$, then the subset of R^k defined by $\neg\varphi(\overline{x}, \overline{a})$ is $\breve{B}$.*

This follows from Claim 1.

Claim 8: *let $\varphi(\overline{x}, \overline{a})$ and $\phi(\overline{x}, \overline{a})$ be formulas with free variables $\overline{x} = (x_1, \dots, x_k)$ and parameters $\overline{a} \in R_0^m$. Let A be the subset of R_0^k defined by $\varphi(\overline{x}, \overline{a})$, and B be the subset of R_0^k defined by $\phi(\overline{x}, \overline{a})$. If the subset of R^k defined by $\varphi(\overline{x}, \overline{a})$ is $\breve{A}$ and the one defined by $\phi(\overline{x}, \overline{a})$ is $\breve{B}$, then the subset of R^k defined by $\varphi(\overline{x}, \overline{a}) \wedge \phi(\overline{x}, \overline{a})$ is $\overset{\smile}{\widehat{A \cap B}}$.*

This follows from Claim 2.

Conclusion: *it follows from Claims 5, 6, 7 and 8, and from Remark 2.2, that the structure $\mathcal{M}$ is an elementary extension of $\mathcal{M}_0$.* □

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